PHYSICS 222, Fall 1995
Solutions for Written Problems
Exam 1, Monday, September 25, 1995, 8:00– 9:15 pm

Problem 1: A. Write down the Biot-Savart law for the magnetic field due to a segment of a current-carrying conductor. Draw an appropriate figure and describe in words each quantity in the law. (3 pts.)

Solution: You can either write down Biot-Savart's law for a current element (short piece of wire) in its vector form

$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{\vec{ds} \times \hat{r}}{r^2}$$

or in its scalar form

$$dB = \frac{\mu_0}{4\pi} I \frac{ds \sin\theta}{r^2}$$

The meaning of each quantity is as follows:

- μ_0 is the permeability constant $4\pi \times 10^{-7}$ N/A^2.
- π is 3.14....
- $d\vec{B}$ is the magnetic field created by the current element. dB is its magnitude.
- The magnitude of $\vec{ds}$ is the length of the current element; it points in the direction of the current flow.
- I is the current flowing through the current element.
- r is the distance between the current element and the position where the magnetic field is calculated. $\hat{r}$ is a unit vector pointing from the current element to this position.
- θ is the angle between $\vec{dl}$ and $\hat{r}$.

Serway, Fig. 19.12 is the appropriate figure.

B. Write down Ampere's law. Draw an appropriate figure and describe in words each quantity in the law. (3 pts.)

Solution: The integral over $\vec{B} \cdot \vec{ds}$ is equal to μ_0 times the enclosed current for any closed Amperean loop, in symbols

$$\oint \vec{B} \cdot \vec{ds} = \mu_0 I_{\text{enclosed}}.$$

Here, $\vec{B}$ is the magnetic field, I_{enclosed} the enclosed current, μ_0 the permeability constant $4\pi \times 10^{-7}$ N/A^2, and $\vec{ds}$ the (infinitesimal) line element of the Amperean loop. Fig. 19.21 in Serway shows an appropriate Amperean loop for a toroid, 19.19 for a thick wire, and 19.24 for a long solenoid.

C. Consider the torus at right, which has inner and outer radii of r_1 and r_2, respectively, and is wrapped by N turns of a wire carrying a current I. Using either the Biot-Savart law or Ampere's law (specify which), show that the field B outside the torus (i.e., at $r < r_1$ or $r > r_2$) is zero. Describe and draw the magnetic field lines inside the torus (i.e., at $r_1 < r < r_2$). (5 pts.)

Solution: The solution for this problem is described in Example 19.7 on pages 557/558 in Serway's textbook: The net current threaded by any cicular path lying outside the toroidal coil is zero (including the "hole in the doughnut"). Therefore, by Ampere's law, the integral $\oint \vec{B} \cdot \vec{ds}$ is zero for all such Amperean loops. If an integral is zero, it is usually not possible to make statements about the value of the function. Here, however, this is different, since the integral is zero for all possible integration paths. If a (continuos) function is zero for all possible integration boundaries, then the function itself has to be indentical to zero. Therefore, in this case, the magnetic field $\vec{B}$ is zero at any point outside the torus. (Note that this is an approximation assuming that the turns are closely spaced, i.e., N is large.) Inside the torus, the field lines are circles in the plane of the torus (clockwise). The magnetic field strength varies as $1/r$.

D. Consider a pair of coils (known as a Helmholtz pair), each of N turns and radius R, and separated by $2R$, shown on the right. (a) Write the expression for the field due to each coil at points along its axis. Which law is used to derive this expression? (b) Find the simplest expression for the magnetic field at the midpoint P between the coils when the currents in the coils are in the same direction and when they are in opposite directions. You may use any formula on the formula sheet as a starting point. (5 pts.)

Solution: (a) Using the Biot-Savart law, one can show that the field due to one coil at points along the axis is

$$B(x) = \frac{\mu_0 N I R^2}{2\left(x^2+R^2\right)^{3/2}},$$

where x is the distance from the center of the loop (you may take this equation from the formula sheet).

(b) At the midpoint P, the field due to one coil is

$$B(R) = \frac{\mu_0 N I R^2}{2\left(R^2+R^2\right)^{3/2}} = \frac{\mu_0 N I}{2R\,(2)^{3/2}}.$$

If the currents in the two loops are antiparallel, the total magnetic field is zero. If the currents are parallel, then the fields add up, and the total magnetic field is

$$B(R) = \frac{\mu_0 N I}{(\sqrt{2})^3 R},$$

pointing along the axis.

Problem 2: In the circuit at right, the antenna receives radio signals and provides the applied (source) voltage. R=10 Ω and L=0.5 mH.

A. If C can be varied from 20 pF to 320 pF, what are the minimal and maximal frequencies to which the circuit can be tuned? Remember that $f = \omega/2\pi$. (4 pts.)

Solution: We know that $f = 1/\left(2\pi\sqrt{LC}\right)$. For C=20 pF, we obtain f=1592 kHz, and for C=320 pF, we have 397 kHz. Therefore, the circuit can be tuned from 397 kHz to 1592 kHz.

B. The amplitude of the signal received from WOI at 1040 kHz is 10 mV. When the circuit is tuned to WHO, what is the amplitude of the current due to that station? (4 pts.)

Solution: The current amplitude is given by the voltage amplitude divided by the impedance of the circuit. At resonance, the impedance is equal to the resistance R. Therefore, the current amplitude is

$$I_m \;=\; V_m/Z \;=\; V_m/R \;=\; 10\ \text{mV}/10\ \Omega \;=\; 1\ \text{mA}.$$

C. The circuit remains tuned to WHO. The amplitude of the signal received from WOI at 640 kHz is also 10 mV. What is the amplitude of the current due to WOI? (4 pts.)

Solution: Here, we have to calculate the impedance for ω=$2\pi\times$640 kHz=4.02×10^6 rad/s. The impedance is $Z=\sqrt{R^2+(X_L-X_C)^2}$, with $X_L=\omega L$ and $X_C=(\omega C)^{-1}$. Before we can find this, we have to calculate the value of the capacitance C, for which the circuit is tuned to WHO (1040 kHz):

$$1040\ \text{kHz}\times 2\pi \;=\; \frac{1}{\sqrt{LC}}$$

Therefore,

$$C \;=\; \frac{1}{(2\pi f)^2 L} \;=\; \frac{1}{(2\pi\times 1040\text{kHz})^2\times 0.5\text{mH}} \;=\; 46.8\text{pF}.$$

We find X_L=4.02×10^6 rad/s$\times$0.5 mH=2010 Ω and X_C=1/(4.02×10^6 rad/s$\times$46.8 pF)=5315 Ω Therefore, the impedance is

$$Z \;=\; \sqrt{(10\ \Omega)^2+(2010\ \Omega-5315\ \Omega)^2} \;=\; 3305\ \Omega$$

The current amplitude is given by the voltage amplitude (10 mV) divided by the impedance (3305 Ω), which gives 3.03 μA.

D. A technical difficulty abruptly shuts off the transmitter at WHO. How long does it take the amplitude of the current oscillating at 1040 kHz to decrease to 10^-3 of its initial value? (4 pts.)

Solution: The amplitude of the current oscillations in a damped oscillator is

$$I(t) \;=\; I_m \exp\left(-Rt/2L\right)\sin\left(\omega t+\phi\right).$$

As we see, the time constant for the decay of the current in the RLC circuit is $\tau=2L/R$, which is 1 mH/10 Ω=100 μs. How many times constants does it take for the current to decrease by 10^{-3}?

$$\begin{aligned} 10^{-3} &= \exp\left(-t/\tau\right).\\ -4.60 = \ln\ 10^{-3} &= -t/\tau.\\ t &= 4.60\tau. \end{aligned}$$

Therefore, after 4.6 time constants (460 μs), the current drops to 10^{-3} of its initial value.

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